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# Dynamics of Neural Networks

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**ABSTRACT.** We propose to study the dynamics of McCulloch-Pitts' neural network and general Boolean networks at the most fundamental level. We propose to study Hopfield's theorem for chaotic iteration and its application to pattern recognition. We propose to furnish a mathematical model of Hebb's postulate of learning. We propose to study the Jacobian problem for Boolean networks.

## 1. Introductory remarks

In 1943, the neurophysiologist W. McCulloch and a Mathematician W. Pitts[6] claimed that the brain could be modeled as a network of logical operations such as *and*, *or*, *not*, and so forth. It had been a revolutionary idea at the time, and had proved to be immensely influential. McCulloch-Pitts model was the first example of what now call a *neural network*. It was the first attempt to understand mental activity as a form of information processing—an insight that provided the inspiration for *artificial intelligence* and *cognitive psychology*. McCulloch-Pitts model was the first indication that a network of very simple logic gates could perform exceedingly complex computation—an insight that was soon incorporated into the general theory of computing machines. McCulloch-Pitts' paper influenced von Neumann to use idealized switch-delay elements derived from the McCulloch-Pitts neuron in the construction of the EDVAC(Electronic Discrete Variable Automatic Computer) (see Aspray and Burks[1]).

In this note, we propose to study the dynamics of McCulloch-Pitts' neural network and general Boolean networks at the most fundamental level.

## 2. McCulloch-Pitts neural network and its dynamics

In a nervous system, each neuron exhibits an impulse of one electric state, called *action potential*. The state of each neuron can be distinguished by the existence and nonexistence of an action potential. Suppose

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that the nervous system consists of  $n$  neurons, we can identify each neuron with an element of  $\{1, 2, \dots, n\}$ . The state of the nervous system at time  $t$  is expressed by a point  $x(t) = (x_1(t), \dots, x_n(t))$  in  $\{0, 1\}^n$ , the set of all 01-strings of length  $n$ . The neural network of McCulloch and Pitts is formulated as follows.

A Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  is called a *threshold function* if there exist  $a \equiv (a_1, \dots, a_n) \in \mathbb{R}^n$  and a threshold value  $\alpha \in \mathbb{R}$  such that  $f(x) = \text{Hev}(\langle a, x \rangle - \alpha)$ , where  $\text{Hev}(u)$  is the Heaviside function. Thus  $f$  is a threshold function if the sets  $f^{-1}(1)$ ,  $f^{-1}(0)$  can be separated by a hyperplane in  $\mathbb{R}^n$ . A Boolean function  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$  is a *threshold function* if each  $f_i$  is threshold, i.e. there exist  $A \in M_n(\mathbb{R})$  and  $b \in \mathbb{R}^n$  such that

$$F(x) = \text{Hev}(Ax - b).$$

The finite state space  $\{0, 1\}^n$  is given a metric structure by the *Hamming metric*  $\rho_H(\cdot, \cdot)$ , i.e.

$$\rho_H(x, y) \equiv \#\{i; x_i \neq y_i\}.$$

Let  $(i, j)$  denote a *synapse*, where  $i, j \in \{1, \dots, n\}$ , neuron  $i$  being the *postsynaptic* neuron and neuron  $j$  being the *presynaptic* neuron. Each entry  $w_{ij}$  expresses the *efficiency* of the synapse  $(i, j)$  and  $b_i$  expresses the *threshold value* for the action potential of neuron  $i$ . The matrix  $A = (a_{ij})$  is called the *synaptic matrix*. Thus the McCulloch-Pitts neural network ( $\equiv$  threshold automata) is described by

$$F(x(t)) = \text{Hev}(Ax(t) - b) \quad (t = 0, 1, \dots).$$

There are two important iteration modes to *implement* the updating of the states on the network: *parallel* and *sequential*.

For parallel iteration mode (*synchronous iteration mode*)

$$x_i(t+1) = \text{Hev}\left[\sum_{j=1}^n a_{ij}x_j(t) - b_i\right] \quad (i = 1, \dots, n) \quad (t = 0, 1, \dots),$$

Gole[2] proved the following:

**Theorem** (Gole). *For parallel iteration mode, the attractors of a symmetrical network of threshold automata are cycles with length  $\leq 2$ .*

For sequential iteration mode (*asynchronous iteration mode*):

$$x_i(t+1) = \text{Hev}\left[\sum_{j=1}^{i-1} a_{ij}x_j(t+1) + \sum_{j=i}^n a_{ij}x_j(t) - b_i\right] \quad (i = 1, 2, \dots, n) \quad (t = 0, 1, \dots),$$

Hopfield[4] proved the following:

**Theorem (Hopfield).** *For Gauss-Seidel iteration, the attractors of a symmetrical network of threshold automata are cycles with nonnegative diagonal entries are fixed points.*

Hopfield's theorem is seminal not only for his result about the asynchronous iteration of McCulloch-Pitts' neural network but open a window onto the essence of *Pattern Recognition* and lead to a solution of *Traveling Salesman Problem*[5].

### 3. Hopfield's theorem for Chaotic iteration

We now study Hopfield's theorem for chaotic iteration, an asynchronous iteration scheme in *Discrete iteration* may go back to 1995 work of Robert[7]. To formulate the problem, let  $F : \{0,1\}^n \rightarrow \{0,1\}^n$ ,  $F = (f_1, \dots, f_n)$ . For  $x \in \{0,1\}^n$ ,  $i = 1, \dots, n$ , let

$$F_i(x) \equiv \begin{pmatrix} x_1 \\ \vdots \\ f_i(x) \\ \vdots \\ x_n \end{pmatrix}.$$

Let  $F(x) \equiv \text{Hev}(Ax - \theta)$  be a threshold map from  $\{0,1\}^n$  into itself, and  $N \equiv \{1, \dots, n\}$ . For  $m = 1, 2, \dots$ , let

$$N^m \equiv \{i_1, \dots, i_m; \quad i_j \in N, j = 1, \dots, m\}$$

be the set of array of  $m$  integers in  $N$ . Let

$$S \equiv \{h_0; h_1; h_2; \dots\},$$

where  $h_j \equiv j_1, j_2, \dots, j_{l(j)} \in N^{l(j)}$  ( $j = 0, 1, \dots$ ). We call  $S$  a *strategy-set*. The *chaotic iteration* for  $F$  with strategy-set  $S = \{h_0; h_1; h_2; \dots\}$  is defined by

$$H_t(x(t)) = x(t+1) \quad (t = 0, 1, \dots), \quad (*)$$

where

$$H_t = F_{t_{l(t)}} \circ F_{t_{l(t)}-1} \circ \dots \circ F_{t_2} \circ F_{t_1}.$$

If  $H_t = F_n \circ F_{n-1} \circ \dots \circ F_1$  ( $t = 0, 1, \dots$ ), then  $(*)$  becomes the Gauss-Seidel iteration. If  $\pi$  is a permutation on  $N$  and  $H_t = F_{\pi(n)} \circ F_{\pi(n-1)} \circ \dots \circ F_{\pi(1)}$  ( $t = 0, 1, \dots$ ), then  $(*)$  is called Gauss-Seidel type iteration. The

chaotic iteration with strategy-set can produce behaviors that are essentially unpredictable. A point  $\xi$  is said to be a *point attractor* for a chaotic iteration with a strategy-set if there exists an initial point  $x(0) \in \{0, 1\}^n$  and  $T \geq 0$  such that  $H_t(x(t)) = x(t+1) = \xi$  for all  $t \geq T$ . For chaotic iteration, a point attractor could not be a fixed point and a cycle might be a non-simple cycle, as the following example shows.

**Example.** Define a threshold map and a strategy-set as follows:

$$F(x) \equiv \text{Hev} \left( \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x - \begin{pmatrix} -0.5 \\ -0.5 \\ -0.5 \end{pmatrix} \right),$$

$S_1 = \{1, 2; 3, 3; 1, 1, 2, 3, 3; 3, 3, 1, 1; 2, 2; 2, 2, 2, 2, 2, 2, 2, 2; \dots\}$ . Then  $F_2 \circ F_1(010) = 110$ ,  $F_3 \circ F_3(110) = 110$ ,  $F_3 \circ F_3 \circ F_1(110) = 110$ ,  $F_1 \circ F_1 \circ F_3 \circ F_3(110) = 110$ ,  $F_2(110) = 110, \dots$ . Thus 110 is a point attractor for this chaotic iteration with strategy-set  $S_1$ . However,  $F(110) \neq 110$ . On the other hand, with another strategy-set  $S_2 = \{1; 3; 3; 1; 1; 3; 3; 1; 1; 3; 3; 1; 1; 3; 3; 1; 1; 3; 3; \dots\}$ , we have  $F_1(010) = 110$ ,  $F_3(110) = 111$ ,  $F_3(111) = 110$ ,  $F_1(110) = 010$ ,  $F_1(010) = 110$ ,  $F_3(110) = 111$ ,  $F_3(111) = 110$ ,  $F_1(110) = 010$ . The question now under consideration is how to get order out of disorder, complexity from simplicity. Now, let us introduce the following notion. The strategy-set is said to be *regular* if there exists a subset  $\{k_0, k_1, \dots, k_{2^n}\} \subset \mathbb{N} \cup \{0\}$  with

$$0 = k_0 < k_1 < \dots < k_{2^n}$$

such that the resulting set  $\{h_{k_j}, h_{k_{j+1}}, \dots, h_{k_{j+1}-1}\}$  is equal to  $\{1, 2, \dots, n\}$  ( $j = 0, 1, \dots, 2^n - 1$ ). Thus  $S_1 = \{1, 2, 3; 1, 2, 3; 1, 2, 3; 1, 2, 3; \dots\}$  and  $S_2 = \{1, 2, 3; 3; 2, 1; 1, 1; 3; 2; 2, 1, 3; 1, 2, 2; 3; 2; 2, 1; 3; 3, 2, 1; 1, 2; 3; 2; 2; 2; 2; 2; \dots\}$  are regular strategy-sets, and  $S_3 = \{1; 2; 3; 1, 2, 3; 1, 2, 3; 1; 2; 1; 2; 1, 2; 1, 2; 1; 1; 1; 1; \dots\}$  is an irregular strategy-set.

Both parallel and sequential iteration are powerful for both brains and artificial networks. The difference between Gauss-Seidel iteration and chaotic iteration is like the difference between the regular repetition of a pattern and the rich, coherent variation.

With the notations and notions stated above, we have the following result of Shih and Tsai[10].

**Theorem 1 .** If  $A = (a_{ij})$  is symmetric with  $a_{ii} \geq 0$  ( $i = 1, 2, \dots, n$ ) and  $S = \{h_0; h_1; \dots\}$  is a regular strategy-set with  $h_i \in N^{l(i)}$  ( $i = 0, 1, \dots$ ), then the sequence  $\{x(t)\}$  generated by  $(*)$  converges to a fixed point of  $F$  for any initial point  $x(0) \in \{0, 1\}^n$ .

**Problem :** Determine the optimality of the length of regular strategy-set.

By Hopfield's theorem, we have the following existence of fixed points.

**Theorem 2 .** If  $A = (a_{ij})$  is symmetric with  $a_{ii} \geq 0$  ( $i = 1, 2, \dots, n$ ), then  $F(x) \equiv \text{Hev}(Ax - \theta)$  has a fixed point.

**Problem:** Does there exist a combinatorial lemma which serves as a basis for the direct proof of Theorem 2?

#### 4. Inverse Dynamics Problem

Let us begin with the following general inverse problem[11]:

"Given an arbitrary set of configurations in  $\{0, 1\}^n$ , is it possible to construct a network of automata for which this set of configurations is the set of attractors?"

Applying Theorem 1, we have the following partial solution of above problem.

**Theorem 3 .** Let  $M \equiv \{\xi^{(1)}, \dots, \xi^{(m)}\}$  be a set of configurations in  $\{0, 1\}^n$  such that  $M$  is orthogonal (i.e.  $\langle \xi^{(i)}, \xi^{(j)} \rangle = 0$  ( $i \neq j$ ),  $\xi^{(i)} \neq 0 \forall i$ ). Let  $A \equiv \sum_{i=1}^m \xi^{(i)} \xi^{(i)T}$ ,  $b \equiv (\frac{1}{2}, \dots, \frac{1}{2})^T$  and  $F(x) \equiv \text{Hev}(Ax - b)$ . Then  $\xi^{(i)}$  ( $i = 1, \dots, m$ ) are fixed points of  $F$  and  $\xi^{(i)}$  ( $i = 1, \dots, m$ ) are point attractors under chaotic iteration with regular strategy-set.

Theorem 3 has two defects. First, we do not know the domains of attractions. Second, orthogonality of  $M$  is required. To overcome these defects, let us propose another solution.

Our new solution is based on Hebb's postulate of learning[3]. Quoting from Hebb's book ([3], p.62):

When an axon of cell  $A$  is near enough to excite a cell  $B$  and repeatedly or persistently takes part in firing it, some growth process or metabolic changes take place in one or both cells such that  $A$ 's efficiency as one of the cells firing  $B$ , is increased.

We now propose to furnish a mathematical model which may be viewed as a mathematical model of Hebb's postulate of learning by the following construction[11].

Let  $M \equiv \{\xi^{(1)}, \dots, \xi^{(m)}\}$  be an orthogonal set in  $\{0, 1\}^n$ ,  $A \equiv \sum_{i=1}^m \xi^{(i)} \xi^{(i)T}$ ,  $b \equiv (\frac{1}{2}, \dots, \frac{1}{2})^T$ , and  $F(x) \equiv \text{Hev}(Ax - b)$ . Define

$$(0, 1)\text{-span}(M) \equiv \left\{ \sum_{i=1}^m \alpha_i \xi^{(i)} ; \alpha_i \in \{0, 1\} \right\},$$

$$U_i \equiv \{x \in \{0, 1\}^n ; 0 < x \leq \xi^{(i)}\} (i = 1, \dots, m),$$

$$x \leq y \iff x_i \leq y_i \quad (i = 1, \dots, n),$$

$$x < y \iff x \leq y \text{ and } x \neq y,$$

$$U_0 \equiv (0,1)\text{-span}\{e_i ; \langle e_i, \xi^{(j)} \rangle = 0 \ (j = 1, \dots, m), i=1,2, \dots, n\} ; (0,1)\text{-span}\{\emptyset\} = \{0\}.$$

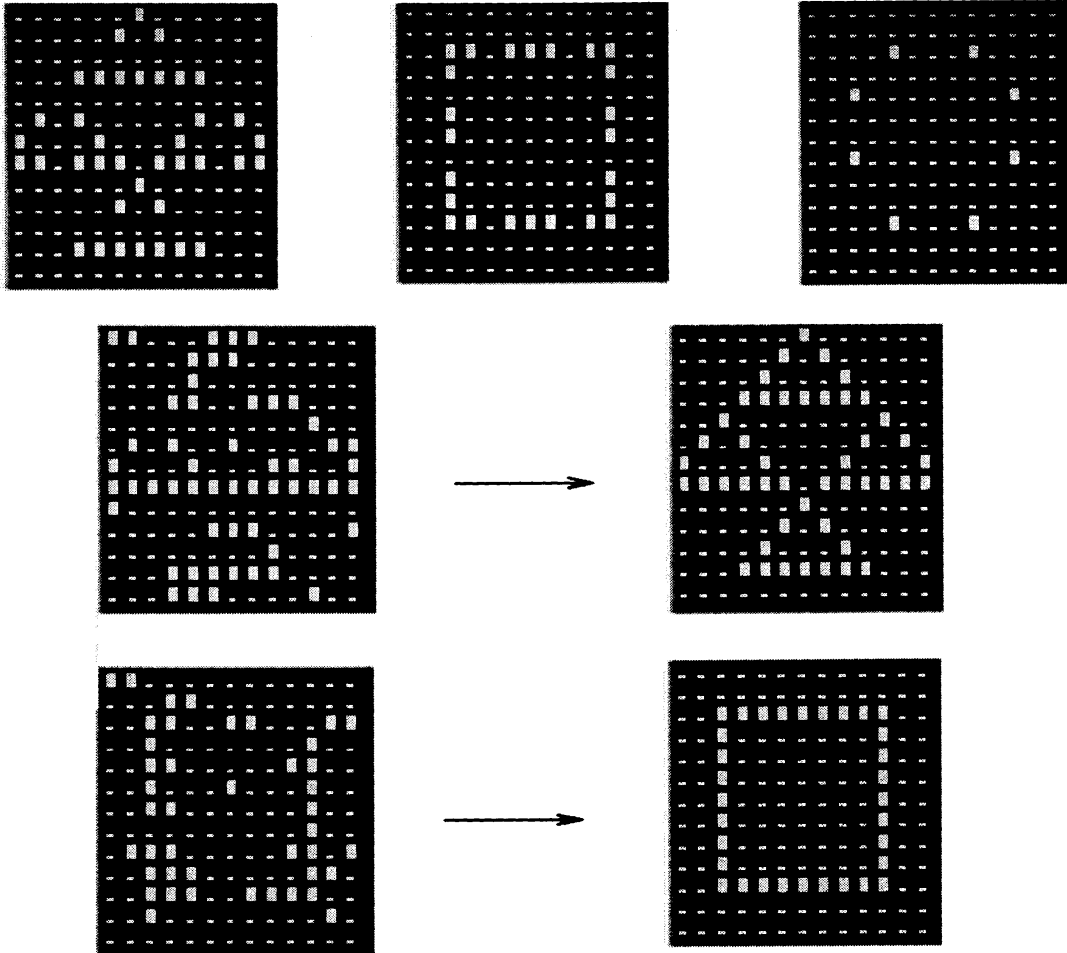
We list some elementary results.

**Fact.**

1.  $\#(0,1)\text{-span}(M)=2^m$ ,
2.  $\{\{\sum_{i=0}^m \alpha_i U_i\} ; \alpha_0 = 1, \alpha_i \in \{0, 1\}, i=1, \dots, m\}$  is a partition of  $\{0, 1\}^n$ ,
3. For all  $x \in (0,1)\text{-span}(M)$ ,  $F(x) = x$ ,
4.  $F$  has a unique fixed point in  $\sum_{i=0}^m \alpha_i U_i$  ( $\alpha_0 = 1, \alpha_i \in \{0, 1\}, i=1, \dots, m$ ).

**Theorem 4** . For each  $x \in U_0 + \sum_{i=1}^m \alpha_i U_i$  ( $\alpha_i \in \{0, 1\}, i=1, \dots, m$ ),  $F(x) = \sum_{i=1}^m \alpha_i \xi^{(i)}$ .

As an illustration of Theorem 4, we have the following computer experiments.



## 5. Dynamics of Boolean networks

Let us begin with some notions and notations. Let  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$ . For  $x \in \{0, 1\}^n$ , denote by  $F'(x)$  and  $\rho(F'(x))$  the Boolean derivative of  $F$  evaluated at  $x$  (i.e.  $F'(x) = (f_{ij}(x))$ , where  $f_{ij}(x) = 1$  if  $f_i(x) \neq f_i(\tilde{x}^j)$ ,  $f_{ij}(x) = 0$  otherwise, here  $\tilde{x}^j = x_1 \dots \bar{x}_j \dots x_n$ ) and the spectral radius of  $F'(x)$ , respectively.

Motivated by the well-known Markus-Yamabe problem about global stability of dynamical system, Shih and Ho[8] proved that

**Theorem 5 .** *Let  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$ . If  $\rho(F'(x)) = 0$  for all  $x \in \{0, 1\}^n$ , and  $F'(x)$  has at most one 1 in each column for all  $x \in \{0, 1\}^n$ , then  $F$  has a unique fixed point  $\xi$  and there exists  $p \leq 2^n$  such that  $F^p(x) = \xi$  for any  $x \in \{0, 1\}^n$ .*

Motivated by the long-standing Jacobian conjecture, Shih and Ho[8] made the following conjecture.

**Combinatorial Fixed Point Conjecture.** *If the cube mapping  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$  is such that  $\rho(F'(x)) = 0$  for all  $x \in \{0, 1\}^n$ , then  $F$  has a unique fixed point.*

Recently Dong and Shih[9] proved this conjecture. Our approach to this conjecture is to make a coherent behavior in the whole cube by way of understanding collective behavior in the subcubes. Now let us introduce the following notation. Let  $x \in \{0, 1\}^n$ . For each  $k = 1, 2, \dots, n-1$  and for each choice of  $k+1$  distinct integers  $i_1, \dots, i_{k+1}$  (which are arranged in any order) from  $\{1, \dots, n\}$ , we define

$$x[\{i_1, \dots, i_k\} \mid i_{k+1}] \equiv \{y \in \{0, 1\}^n; y_{i_{k+1}} = x_{i_{k+1}}, y_j = x_j \text{ for all } j \neq i_1, \dots, i_k\}.$$

The following lemma will play a prominent role in the proof of the conjecture. And the kernel of the lemma reveals an unexpected regularity hidden in the spectral condition.

**Lemma.** *Let  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$ . If  $\rho(F'(x)) = 0$  for all  $x \in \{0, 1\}^n$ , then for each  $x \in \{0, 1\}^n$  and for each  $k = 1, \dots, n-1$  and for each choice of  $k+1$  distinct integers  $i_1, \dots, i_{k+1}$  (which are arranged in any order) from  $\{1, \dots, n\}$ , there exists a unique point  $\alpha \in x[\{i_1, \dots, i_k\} \mid i_{k+1}]$  such that  $f_j(\alpha) = \alpha_j$  for all  $j = i_1, \dots, i_k$ .*

For a long time we intend to prove that such  $p$  in Theorem 5 should be lesser or equal to  $n$  (i.e. the optimal transient length of  $\xi$  is  $n$ ) by establishing the following conjecture : *Conditions “ $\rho(F'(x)) = 0$  for all  $x \in \{0, 1\}^n$ ”, and “ $F'(x)$  has at most one 1 in each column for all  $x \in \{0, 1\}^n$ ” together imply that  $\sup_{x \in \{0, 1\}^n} \{F'(x)\}$  contains a zero row.* Note that the condition “ $F'(x)$  has at most one 1 in each column for all  $x \in \{0, 1\}^n$ ” is equivalent to the condition “ $F$  is a nonexpansive map with respect to Hamming metric, i.e.



$\rho_H(F(x), F(y)) \leq \rho_H(x, y)$  for all  $x, y \in \{0, 1\}^n$ .” Recently Dong and Shih[9] proved that if  $n \leq 4$  the answer to the conjecture is affirmative. The proof is based on the above lemma. The case  $n \geq 5$  of the conjecture remains open.

Let us mention in conclusion that the spectral condition “ $\rho(F'(x)) = 0$  for all  $x \in \{0, 1\}^n$ ” implies that  $F$  leaves a unique point invariant. And on toward microscopic perspectives: the spectral condition also implies that for each  $k = 1, \dots, n-1$  and for each  $k$ -subcube the boolean function  $F$  leaves a unique point in the  $k$ -subcube having  $k$  components invariant in a very regular pattern indeed. This phenomenon is of exceptional interesting feature, perhaps because we can easily find a boolean function  $F : \{0, 1\}^n \rightarrow \{0, 1\}^n$  leaves a unique point invariant but there exists a  $k$ -subcube for which  $F$  does not leave a point in the  $k$ -subcube having  $k$  components invariant.

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